

AN ANALOGUE OF RADFORD'S S^4 -FORMULA FOR FINITE TENSOR CATEGORIES

PAVEL ETINGOF, DMITRI NIKSHYCH, AND VIKTOR OSTRIK

ABSTRACT. We develop the theory of Hopf bimodules for a finite rigid tensor category \mathcal{C} . Then we use this theory to define a distinguished invertible object D of \mathcal{C} and an isomorphism of tensor functors $\delta : V^{**} \rightarrow D \otimes^{**} V \otimes D^{-1}$. This provides a categorical generalization of Radford's S^4 formula for finite dimensional Hopf algebras [R1], which was proved in [N] for weak Hopf algebras, in [HN] for quasi-Hopf algebras, and conjectured in general in [EO]. When \mathcal{C} is braided, we establish a connection between δ and the Drinfeld isomorphism of \mathcal{C} , extending the result of [R2]. We also show that a factorizable braided tensor category is unimodular (i.e. $D = \mathbf{1}$). Finally, we apply our theory to prove that the pivotalization of a fusion category is spherical, and give a purely algebraic characterization of exact module categories defined in [EO].

1. INTRODUCTION

One of the most important general results about finite dimensional Hopf algebras is Radford's formula for the forth power of the antipode. This formula reads: for any finite dimensional Hopf algebra H over a field k with antipode S ,

$$(1) \quad S^4(h) = a^{-1}(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})a,$$

where a and α are the distinguished grouplike elements of H and H^* , respectively.

Recently, several generalizations of this formula were discovered. Namely, in [N] the formula was extended to weak Hopf algebras, and in [HN] to quasi-Hopf algebras. Finally, in [EO] the authors conjectured a generalization of Radford's formula to any finite tensor category (Conjecture 2.15). One of the main achievements of the present paper is a proof of this conjecture.¹

More specifically, we show in Theorem 3.3 (which is our main theorem) that for a finite tensor category \mathcal{C} there is an invertible object D , called the *distinguished invertible object* of \mathcal{C} , and an isomorphism of tensor functors $\delta : ?^{**} \cong D \otimes^{**} ? \otimes D^{-1}$. We also show that our definition of the distinguished object is the same as in [EO, Definition 2.12], which means that our main theorem implies Conjecture 2.15 from [EO].

Further, we apply the main theorem to braided tensor categories to show that a factorizable braided category is unimodular (in particular, the center $\mathcal{Z}(\mathcal{C})$ of a finite tensor category \mathcal{C} is unimodular). We also relate the isomorphism δ with the Drinfeld isomorphism $u : ? \cong ?^{**}$.

Date: April 27, 2004.

¹We note that this result is, essentially, equivalent to the statement that Radford's formula holds for weak quasi-Hopf algebras. However, weak quasi-Hopf algebras are very cumbersome objects, and we avoid working with them by using a much simpler purely categorical language.

In the case when \mathcal{C} is semisimple, we describe the isomorphism δ in terms of Müger's squared norms of simple objects, and show that the pivotalization of \mathcal{C} is spherical (this was previously known only in characteristic 0 [ENO]).

The paper is organized as follows. In Section 2 we define a category \mathcal{H} of Hopf bimodules over \mathcal{C} as the category of right modules over the algebra $A = \underline{\text{Hom}}(\mathbf{1}, \mathbf{1})$ in $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$. We define a tensor category structure on \mathcal{H} and prove in Proposition 2.3 that there is a tensor equivalence between \mathcal{C} and \mathcal{H} (which establishes a categorical version of the Fundamental Theorem for Hopf bimodules). We use this result as a major tool in our arguments throughout the paper.

In Section 3 we define the *distinguished invertible object* of \mathcal{C} as the unique up to an isomorphism object D such that $A^* \cong (D \boxtimes \mathbf{1}) \otimes A$, following the idea used in [HN] for quasi-Hopf algebras. We use the category of Hopf bimodules to construct in Theorem 3.3 a natural tensor isomorphism $\delta : ?^{**} \cong D \otimes ?? \otimes D^{-1}$.

In Sections 4 and 5 we work with a braided finite tensor category \mathcal{C} . We introduce a categorical notion of *factorizability* of \mathcal{C} and show that the center of a finite tensor category is factorizable. We prove that a factorizable tensor category is automatically unimodular (i.e., $D \cong \mathbf{1}$), extending the results known for (weak, quasi) Hopf algebras. We also show that in the case of unimodular \mathcal{C} one has $\delta = u^{-1} \circ u^*$ where $u : ? \cong ?^{**}$ is the Drinfeld isomorphism.

In Section 6 we check that our definition of D agrees with the one given in [EO], that is, the projective cover of the unit object $\mathbf{1}$ coincides with the injective hull of D .

In Section 7 we specialize to the case when \mathcal{C} is semisimple and give a convenient numerical characterization of the isomorphism δ in Theorem 7.3. As a consequence, we obtain that the pivotalization of a semisimple category is spherical.

The appendix contains an algebraic characterization of exact module categories studied in [EO]. We show in Theorem 8.1 that if \mathcal{M} is a module category over \mathcal{C} , B is a k -algebra such that $\mathcal{M} \cong B\text{-mod}$, and $\bar{F} : \mathcal{C} \rightarrow \text{Vect}_k$ is the fiber functor constructed from the action of \mathcal{C} on \mathcal{M} , then \mathcal{M} is exact if and only if the algebra $H = \text{End}(\bar{F})$ is a projective $B \otimes B^{\circ}$ -module.

Remark. As was anticipated by G. Kuperberg, the statement and proof of the categorical Radford's formula do not make an essential use of the additive structure of the category \mathcal{C} (and thus can be generalized to the non-additive case). In this respect, they are similar to Theorem 3.10 in [Ku], which gives Radford's formula for a Hopf object in a rigid monoidal (possibly non-additive) category.

Acknowledgements. We are grateful to G. Kuperberg for a useful discussion. The research of P.E. was supported by the NSF grant DMS-9988796. The research of D.N. was supported by the NSF grant DMS-0200202. The research of V.O. was supported by the NSF grants DMS-0098830 and DMS-0111298.

2. AN ANALOGUE OF THE CATEGORY OF HOPF BIMODULES

We work over an algebraically closed field k . All categories considered in this paper are abelian over k with all objects being of finite length and all morphism spaces being finite dimensional.

Such a category \mathcal{C} is said to be *finite* if it has finitely many isomorphism classes of simple objects and every object has a projective cover. That is, \mathcal{C} is equivalent to the category of finite dimensional representations of some finite dimensional k -algebra.

By a *tensor category* over k we understand an abelian rigid monoidal category. We refer the reader to [BK, K, EO] for the general theory of tensor categories and module categories over them.

Recall the notion of Deligne's tensor product $\mathcal{C} \boxtimes \mathcal{D}$ of abelian categories \mathcal{C}, \mathcal{D} , see [D]. By definition, this is the universal object for the functor assigning to every abelian category \mathcal{A} the category of additive right (equivalently, left) exact bifunctors from $\mathcal{C} \times \mathcal{D}$ to \mathcal{A} . If \mathcal{C}, \mathcal{D} are tensor categories then the category $\mathcal{C} \boxtimes \mathcal{D}$ has a natural structure of a tensor category with the tensor product

$$(2) \quad \otimes \times \otimes : (\mathcal{C} \boxtimes \mathcal{D}) \times (\mathcal{C} \boxtimes \mathcal{D}) \rightarrow \mathcal{C} \boxtimes \mathcal{D},$$

(which we will still denote \otimes), and with the unit object $\mathbf{1} \boxtimes \mathbf{1}$.

Deligne's tensor product can also be applied to functors. Namely, if $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $G : \mathcal{D} \rightarrow \mathcal{D}'$ are additive right (left) exact functors between abelian categories then one can define the functor $F \boxtimes G : \mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{C}' \boxtimes \mathcal{D}'$.

Let \mathcal{C} be a finite tensor category over k . Let \mathcal{C}^{op} be the opposite tensor category, that is, $\mathcal{C}^{\text{op}} = \mathcal{C}$ as a category but the tensor product \otimes^{op} in \mathcal{C}^{op} is different: $X \otimes^{\text{op}} Y = Y \otimes X$.

The category \mathcal{C} has a natural structure of an exact module category [EO] over $(\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}, \otimes)$ coming from

$$(3) \quad \otimes \circ (\otimes \times \text{id}_{\mathcal{C}^{\text{op}}}) : \mathcal{C} \times \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}.$$

We will denote this action by $(X, V) \mapsto X \circ V$ ($X \in \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}, V \in \mathcal{C}$).

The internal Hom, $\underline{\text{Hom}}(V_1, V_2)$, of two objects V_1, V_2 in \mathcal{C} is an object of $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ representing the functor $\text{Hom}_{\mathcal{C}}(? \circ V_1, V_2)$ (the latter contravariant functor is left exact, so it is representable). This means that there is a natural isomorphism

$$(4) \quad \text{Hom}_{\mathcal{C}}(X \circ V_1, V_2) \cong \text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}}(X, \underline{\text{Hom}}(V_1, V_2)).$$

We refer to [EO] for the properties of $\underline{\text{Hom}}$. The object $A := \underline{\text{Hom}}(\mathbf{1}, \mathbf{1})$ has a natural structure of an algebra in the category $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$.

Definition 2.1. The category of right A -modules in $(\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}, \otimes)$ will be called the category of *Hopf bimodules* over \mathcal{C} .

Let \mathcal{H} denote the category of Hopf bimodules.

Observe that $(X, Y) \mapsto \text{Hom}_{\mathcal{C}}(*Y, X)$ is an additive bifunctor from $\mathcal{C}^{\text{op}} \times \mathcal{C}$ to Vect_k . Hence it defines an additive functor $H_{\mathcal{C}} : \mathcal{C}^{\text{op}} \boxtimes \mathcal{C} \rightarrow \text{Vect}_k$. Therefore, one can define another tensor product \odot on $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ by

$$(5) \quad (\text{id}_{\mathcal{C}} \boxtimes H_{\mathcal{C}}) \boxtimes \text{id}_{\mathcal{C}^{\text{op}}} : (\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}) \boxtimes (\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}) \cong (\mathcal{C} \boxtimes (\mathcal{C}^{\text{op}} \boxtimes \mathcal{C})) \boxtimes \mathcal{C}^{\text{op}} \rightarrow \mathcal{C} \boxtimes \mathcal{C}^{\text{op}},$$

where we implicitly used a natural action of Vect_k on \mathcal{C} . One checks directly that A is the unit object for \odot .

Remark 2.2. (i) Another way to define \odot is the following: one identifies \mathcal{C}^{op} with the dual category \mathcal{C}^{\vee} via the functor $X \mapsto *X$. Then the category $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ is identified with the category of left exact functors from \mathcal{C} to itself (for example $X \boxtimes Y \in \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ corresponds to the functor $Z \mapsto \text{Hom}(*Z, Y) \otimes X$), and \odot corresponds to the composition of functors. Under this identification the object $A \in \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ representing the functor $X \boxtimes Y \mapsto \text{Hom}(X \otimes Y, \mathbf{1}) = \text{Hom}(X, Y^*)$ corresponds to the identity functor (this is why we use $X \mapsto *X$ and not $X \mapsto X^*$ to identify \mathcal{C}^{op} and \mathcal{C}^{\vee}).

(ii) If \mathcal{C} is the representation category of a Hopf algebra (or quasi-Hopf algebra, or weak Hopf algebra) H then \mathcal{H} is the category of usual H -Hopf bimodules [LS, HN, BNS] and \odot is dual to the usual bimodule tensor product (thus $M \odot N = (N^* \otimes_H M^*)^*$ where the star denotes the dual vector space).

The Fundamental Theorem for Hopf modules over a Hopf algebra H [LS] states that the category of H -Hopf modules is equivalent to the category of vector spaces. In [HN] it was explained that for quasi-Hopf algebras the notion of a Hopf module should be replaced by that of a Hopf *bimodule* and the Fundamental Theorem for Hopf bimodules over a quasi-Hopf algebra [HN, Proposition 3.11] algebra was proved. Proposition 2.3 below provides a categorical version of the Fundamental Theorem for Hopf bimodules and generalizes the results of [LS, HN].

Proposition 2.3. (a) *The category \mathcal{H} is a tensor subcategory of $(\mathcal{C} \boxtimes \mathcal{C}^{op}, \odot)$ and the functors $(? \boxtimes \mathbf{1}) \otimes A$ and $(\mathbf{1} \boxtimes ?) \otimes A$ from \mathcal{C} to \mathcal{H} are equivalences of tensor categories.*

(b) *There is a natural isomorphism of tensor functors*

$$(6) \quad \rho : (? \boxtimes \mathbf{1}) \otimes A \cong (\mathbf{1} \boxtimes ?) \otimes A.$$

Proof. (a) For all objects V in \mathcal{C} we have

$$\begin{aligned} \underline{\text{Hom}}(\mathbf{1}, V) &= \underline{\text{Hom}}(\mathbf{1}, (V \boxtimes \mathbf{1}) \circ \mathbf{1}) \cong (V \boxtimes \mathbf{1}) \otimes \underline{\text{Hom}}(\mathbf{1}, \mathbf{1}) = (V \boxtimes \mathbf{1}) \otimes A, \\ \underline{\text{Hom}}(\mathbf{1}, V) &= \underline{\text{Hom}}(\mathbf{1}, (\mathbf{1} \boxtimes V) \circ \mathbf{1}) \cong (\mathbf{1} \boxtimes V) \otimes \underline{\text{Hom}}(\mathbf{1}, \mathbf{1}) = (\mathbf{1} \boxtimes V) \otimes A. \end{aligned}$$

It follows from [EO], Theorem 3.17 that $(? \boxtimes \mathbf{1}) \otimes A$ is an equivalence between \mathcal{C} and \mathcal{H} . To see that it is tensor, observe that under the identification in Remark 2.2 (i) the functor $(? \boxtimes \mathbf{1}) \otimes A$ (and similarly $(\mathbf{1} \boxtimes ?) \otimes A$) sends $V \in \mathcal{C}$ to the functor $V \boxtimes ?$ from \mathcal{C} to itself. Thus the associativity constraint in the category \mathcal{C} gives rise to a tensor structure on these functors.

(b) The tensoriality of the natural isomorphism (6) is obvious from description in (a). It is also equivalent to commutativity of the following diagram

$$(7) \quad \begin{array}{ccc} ((V \boxtimes \mathbf{1}) \otimes A) \odot ((W \boxtimes \mathbf{1}) \otimes A) & \longrightarrow & ((V \otimes W) \boxtimes \mathbf{1}) \otimes A \\ \downarrow \rho_V \odot \rho_W & & \downarrow \rho_{V \otimes W} \\ ((\mathbf{1} \boxtimes V) \otimes A) \odot ((\mathbf{1} \boxtimes W) \otimes A) & \longrightarrow & (\mathbf{1} \boxtimes (V \otimes W)) \otimes A. \end{array}$$

□

Remark 2.4. Another way to state Proposition 2.3 (b) is to say that the following diagram commutes:

$$(8) \quad \begin{array}{ccc} (V \boxtimes \mathbf{1}) \otimes (W \boxtimes \mathbf{1}) \otimes A & \longrightarrow & ((V \otimes W) \boxtimes \mathbf{1}) \otimes A \\ \downarrow \text{id} \otimes \rho_W & & \downarrow \rho_{V \otimes W} \\ (V \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes W) \otimes A & & (\mathbf{1} \boxtimes (V \otimes W)) \otimes A \\ \downarrow & & \uparrow \\ (\mathbf{1} \boxtimes W) \otimes (V \boxtimes \mathbf{1}) \otimes A & \xrightarrow{\text{id} \otimes \rho_V} & (\mathbf{1} \boxtimes W) \otimes (\mathbf{1} \boxtimes V) \otimes A. \end{array}$$

3. CONSTRUCTION OF AN ISOMORPHISM BETWEEN DUALITY FUNCTORS

Let $A = \underline{\text{Hom}}(\mathbf{1}, \mathbf{1})$ be the algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ defined in the previous Section and let M be a left A -module. Recall [O1] that M^* has a natural structure of a right A -module with the action given by

$$(9) \quad M^* \otimes A \xrightarrow{m_A^* \otimes \text{id}_A} M^* \otimes A^* \otimes A \xrightarrow{\text{id}_{M^*} \otimes \text{coev}_A} M^*,$$

where $m_A : A \otimes M \rightarrow M$ be the left action of A on M and coev_A is the coevaluation morphism of A . In particular A^* has a canonical ² structure of a Hopf bimodule. Thus according to Proposition 2.3 (a) there exists a unique up to an isomorphism object $D \in \mathcal{C}$ such that

$$(10) \quad (D \boxtimes \mathbf{1}) \otimes A \cong A^*$$

as Hopf bimodules. Moreover isomorphism (10) is unique up to scaling. It follows immediately from the definition that the Frobenius-Perron dimension (see [EO]) of D equals to 1 and thus D is invertible.

Definition 3.1. The object D defined by (10) is called a *distinguished invertible object* of \mathcal{C} .

Remark 3.2. Our definition of the distinguished invertible object of \mathcal{C} categorically extends definitions of distinguished group-like element, or modulus, of a Hopf algebra [R1], quasi-Hopf algebra [HN], or weak Hopf algebra [N]. In Section 6 we show that Definition 3.1 is equivalent to [EO, Definition 2.12].

The classical formula of D. Radford [R1] expresses the fourth power of the antipode of a finite-dimensional Hopf algebra H in terms of distinguished group-like elements of H and H^* . The categorical version of this formula below is a main result of this paper.

Theorem 3.3. *Let \mathcal{C} be a finite tensor category. There is a natural isomorphism of tensor functors,*

$$(11) \quad \delta : ?^{**} \rightarrow D \otimes ?? \otimes D^{-1}.$$

Proof. Isomorphism (10) produces a canonical isomorphism of algebras

$$(12) \quad A^{**} = \underline{\text{Hom}}(A^*, A^*) \cong \underline{\text{Hom}}((D \boxtimes \mathbf{1}) \otimes A, (D \boxtimes \mathbf{1}) \otimes A) = (D \boxtimes \mathbf{1}) \otimes A \otimes (D \boxtimes \mathbf{1})^*.$$

We will identify these algebras using this isomorphism.

Recall that we have a tensor isomorphism (6) $\rho_V : (V \boxtimes \mathbf{1}) \otimes A \cong (\mathbf{1} \boxtimes V) \otimes A$. Its double dual $\rho_V^{**} : (V^{**} \boxtimes \mathbf{1}) \otimes A^{**} \cong (\mathbf{1} \boxtimes **V) \otimes A^{**}$ is also tensor (i.e. the diagram analogous to (8) commutes).

Thus we have a tensor isomorphism of right A -modules

$$\tilde{\rho}_V : (V^{**} \boxtimes \mathbf{1}) \otimes (D \boxtimes \mathbf{1}) \otimes A \cong (\mathbf{1} \boxtimes **V) \otimes (D \boxtimes \mathbf{1}) \otimes A$$

defined by $\tilde{\rho}_V \otimes \text{id}_{(D \boxtimes \mathbf{1})^*} = \rho_V^{**}$.

Now define $\tilde{\delta}_V$ as the following composition:

$$\begin{aligned} (V^{**} \boxtimes \mathbf{1}) \otimes A &= ((V^{**} \otimes D^*) \boxtimes \mathbf{1}) \otimes (D \boxtimes \mathbf{1}) \otimes A \xrightarrow{\tilde{\rho}_V \otimes D^*} (\mathbf{1} \boxtimes (**V \otimes D^*)) \otimes (D \boxtimes \mathbf{1}) \otimes A = \\ &= (D \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes (**V \otimes D^*)) \otimes A \xrightarrow{\text{id} \otimes \rho^{**} V \otimes D^*} (D \boxtimes \mathbf{1}) \otimes ((**V \otimes D^*) \boxtimes \mathbf{1}) \otimes A = \end{aligned}$$

²In this paper, “canonical” means that there is a distinguished choice which should be obvious to the reader.

$$= ((D \otimes **V \otimes D^*) \boxtimes \mathbf{1}) \otimes A.$$

Obviously, the isomorphism $\tilde{\delta}_V$ is tensor (again, the diagram analogous to (8) commutes).

Finally, define the isomorphism $\delta_V : V^{**} \rightarrow D \otimes **V \otimes D^*$ by the condition $\delta_V \otimes \text{id}_A = \tilde{\delta}_V$. Since $\tilde{\delta}_V$ is a morphism of right A -modules Proposition 2.3 (a) implies that δ_V is well defined. The fact that $\tilde{\delta}_V$ is tensor translates into the fact that δ_V is an isomorphism of tensor functors. The Theorem is proved. \square

Corollary 3.4. *There is a positive integer N such that the N th powers of tensor functors $?^{**}$ and $^{**}?$ are naturally isomorphic.*

Proof. Since \mathcal{C} has finitely many non-isomorphic invertible objects, there exists N such that $D^{\otimes N} \cong \mathbf{1}$. \square

4. UNIMODULARITY OF FACTORIZABLE CATEGORIES

Definition 4.1. We will say that \mathcal{C} is *unimodular* if its distinguished invertible object is isomorphic to $\mathbf{1}$.

Equivalently, \mathcal{C} is unimodular if A is a self-dual object of $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$.

Let (\mathcal{C}, σ) be a braided finite tensor category, where σ is a natural isomorphism of bifunctors $\sigma : \otimes \cong \otimes^{\text{op}}$ satisfying hexagon axioms [BK, K]. Let $\mathcal{Z}(\mathcal{C})$ be the center of \mathcal{C} . Recall (see e.g. [K]) that the objects of $\mathcal{Z}(\mathcal{C})$ are pairs $(X, e_X(?))$ where $e_X(?)$ is a functorial isomorphism $e_X(Y) : X \otimes Y \simeq Y \otimes X$ defined for all $Y \in \mathcal{C}$ and satisfying certain axioms. Now we define a tensor functor $G : \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \rightarrow \mathcal{Z}(\mathcal{C})$ in the following way:

$$(13) \quad G(X \boxtimes Y) = (X \otimes Y, e_{X \otimes Y})$$

where

$$e_{X \otimes Y}(Z) : X \otimes Y \otimes Z \xrightarrow{\text{id}_X \otimes \sigma_{Z,Y}^{-1}} X \otimes Z \otimes Y \xrightarrow{\sigma_{X,Z} \otimes \text{id}_Y} Z \otimes X \otimes Y.$$

The functor G has a natural structure of a braided tensor functor.

Definition 4.2. We will say that a finite braided tensor category \mathcal{C} is *factorizable* if G is an equivalence of tensor categories.

Remark 4.3. Factorizable Hopf algebras were introduced and studied by N. Reshetikhin and M. Semenov-Tian-Shansky in [RS]. This notion was extended to weak Hopf algebras in [NTV] and to quasi-Hopf algebras in [BT]. One can directly check that our Definition 4.2 extends the previous definitions. E.g., using a computation analogous to the one given by H.-J. Schneider for Hopf algebras in [S, Theorem 4.3] one shows that a weak Hopf algebra H is factorizable if and only if its representation category $\text{Rep}(H)$ is factorizable in the sense of Definition 4.2, so the two definitions agree in this case.

The notion of a factorizable braided tensor category also extends that of a modular category to the case when \mathcal{C} is not necessarily ribbon or semisimple. Indeed, every semisimple finite tensor category is equivalent to the representation category of some semisimple weak Hopf algebra [O1], and it was shown in [NTV] that a semisimple ribbon weak Hopf algebra is modular if and only if it is factorizable.

An example of a factorizable category is given by the center of a tensor category.

Proposition 4.4. *Let \mathcal{C} be a finite tensor category. Then its center $\mathcal{Z}(\mathcal{C})$ is factorizable.*

Proof. Let \mathcal{M} be an exact module category over a finite tensor category \mathcal{D} and let $\mathcal{D}_{\mathcal{M}}^*$ be the dual tensor category, whose objects are \mathcal{D} -module endofunctors of \mathcal{M} , see [EO, O1] for definitions. By [EO, Corollary 3.35] there is a canonical tensor equivalence of tensor categories $Q : \mathcal{Z}(\mathcal{D}) \cong \mathcal{Z}(\mathcal{D}_{\mathcal{M}}^*)$ that assigns to every object in $\mathcal{Z}(\mathcal{D})$ its module action on \mathcal{M} . We will apply this result to the case when $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ and $\mathcal{M} = \mathcal{C}$.

Recall that there is a tensor equivalence $\mathcal{Z}(\mathcal{C}) \cong (\mathcal{C} \boxtimes \mathcal{C}^{\text{op}})_{\mathcal{C}}^*$ [EO, O1]. It is straightforward to check that the tensor functor $G : \mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{C})^{\text{op}} \rightarrow \mathcal{Z}(\mathcal{Z}(\mathcal{C}))$ defined as in equation (13) is the composition of the obvious equivalence $\mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{C})^{\text{op}} \cong \mathcal{Z}(\mathcal{C} \boxtimes \mathcal{C}^{\text{op}})$ and $Q : \mathcal{Z}(\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}) \cong \mathcal{Z}(\mathcal{Z}(\mathcal{C}))$ defined in the previous paragraph. Therefore, G is an equivalence, i.e., $\mathcal{Z}(\mathcal{C})$ is factorizable. \square

Next, we establish a categorical generalization of another Radford's result [R3] stating that a factorizable Hopf algebra is unimodular (see also [BT] for quasi-Hopf algebras).

Proposition 4.5. *If \mathcal{C} is a factorizable tensor category then \mathcal{C} is unimodular.*

Proof. Observe that \mathcal{C} is a $\mathcal{Z}(\mathcal{C})$ -module category via the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$,

$$(14) \quad Z \bullet V := F(Z) \otimes V,$$

for all objects $Z \in \mathcal{Z}(\mathcal{C})$ and V in \mathcal{C} . By the factorizability of \mathcal{C} there is a natural isomorphism of functors $? \bullet \mathbf{1} \cong G^{-1}(?) \circ \mathbf{1}$. Let $I : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ be the induction functor [EO, Lemma 3.38], i.e., the right adjoint functor of F . There is a sequence of natural isomorphisms :

$$\begin{aligned} \text{Hom}_{\mathcal{Z}(\mathcal{C})}(Z, I(V)) &\cong \text{Hom}_{\mathcal{C}}(F(Z), V) \\ &= \text{Hom}_{\mathcal{C}}(Z \bullet \mathbf{1}, V) \\ &\cong \text{Hom}_{\mathcal{C}}(G^{-1}(Z) \circ \mathbf{1}, V) \\ &\cong \text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}}(G^{-1}(Z), \underline{\text{Hom}}(\mathbf{1}, V)) \\ &\cong \text{Hom}_{\mathcal{Z}(\mathcal{C})}(Z, G(\underline{\text{Hom}}(\mathbf{1}, V))). \end{aligned}$$

Hence, $I(?)$ is naturally isomorphic to $G(\underline{\text{Hom}}(\mathbf{1}, ?))$. For all objects V in \mathcal{C} we have $F(*V) = *F(V)$ by the definition of duality in $\mathcal{Z}(\mathcal{C})$, i.e., F commutes with the left dual functor. Therefore the adjoint functor I commutes with the right dual functor. In particular, $I(\mathbf{1})^* = I(\mathbf{1})$ and $\underline{\text{Hom}}(\mathbf{1}, \mathbf{1}) = \underline{\text{Hom}}(\mathbf{1}, \mathbf{1})^*$ (note that the tensor functor G commutes with duality), i.e., \mathcal{C} is unimodular. \square

5. RELATION WITH THE DRINFELD ISOMORPHISM IN THE BRAIDED CASE

Let \mathcal{C} be a braided tensor category with braiding $\sigma : \otimes \cong \otimes^{\text{op}}$. It is well known that in this case there is a natural isomorphism $u : ? \rightarrow ?^{**}$, called the *Drinfeld isomorphism*, given by

$$(15) \quad u_V : V \xrightarrow{\text{coev}_V^*} V \otimes V^* \otimes V^{**} \xrightarrow{\sigma_{V, V^*}} V^* \otimes V \otimes V^{**} \xrightarrow{\text{ev}_V} V^{**},$$

where $\text{coev}_V : \mathbf{1} \rightarrow V \otimes V^*$ and $\text{ev}_V : V^* \otimes V \rightarrow \mathbf{1}$ are the coevaluation and evaluation morphisms attached to an object V .

Let $\mathcal{C} = \text{Rep}(H)$ be the category of representations of a finite-dimensional Hopf algebra H . Then a result of Radford [R2] relates the Drinfeld isomorphism $u_V : V \cong V^{**}$ and the tensor isomorphism $\delta_V : V^{**} \rightarrow D \otimes {}^{**}V \otimes D^{-1}$ constructed in Theorem 3.3. An extension of this result to weak Hopf algebras was obtained in [ENO, Lemma 5.12]. The proofs use Hopf algebra language and techniques. Below we give a categorical generalization of these results (we restrict ourselves to the unimodular case).

Theorem 5.1. *Let \mathcal{C} be a unimodular braided finite tensor category. Let $\delta_V : V^{**} \rightarrow {}^{**}V$ be the tensor isomorphism constructed in Theorem 3.3 and let $u_V : V \rightarrow V^{**}$ be the Drinfeld isomorphism. Then there is an equality of natural isomorphisms*

$$(16) \quad \delta_V = u_{**V}^{-1} \circ u_V^*.$$

Proof. Let $\rho_V : (V \boxtimes \mathbf{1}) \otimes A \cong (\mathbf{1} \boxtimes V) \otimes A$ be as in Theorem 3.3. Recall that by definition,

$$(17) \quad (\delta_V \boxtimes \text{id}_1) \otimes \text{id}_A = \rho_{**V}^{-1} \circ \rho_V^{**}.$$

Let $\Sigma = \sigma \boxtimes \sigma^{-1}$ be the braiding on $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$. Define a natural isomorphism

$$(18) \quad \iota : (? \boxtimes \mathbf{1}) \otimes A \rightarrow ({}^{**} \boxtimes \mathbf{1}) \otimes A$$

as the following composition :

$$(19) \quad \iota_V : (V \boxtimes \mathbf{1}) \otimes A \xrightarrow{\rho_V} (\mathbf{1} \boxtimes V) \otimes A \xrightarrow{\Sigma_{\mathbf{1} \boxtimes V, A}} A \otimes (\mathbf{1} \boxtimes V) \xrightarrow{\rho_V^*} A \otimes (V^{**} \boxtimes \mathbf{1}) \xrightarrow{\Sigma_{A, V^{**} \boxtimes \mathbf{1}}} (V^{**} \boxtimes \mathbf{1}) \otimes A.$$

Observe that in terms of Hom spaces the isomorphism ι_V is given as the following sequence of natural isomorphisms:

$$\begin{aligned} \text{Hom}(X_1 \boxtimes X_2, (V \boxtimes \mathbf{1}) \otimes A) &\cong \text{Hom}(V^* \otimes X_1 \otimes X_2, \mathbf{1}) \\ &\cong \text{Hom}(X_1 \otimes V^* \otimes X_2, \mathbf{1}) \\ &\cong \text{Hom}(X_1 \otimes X_2 \otimes V^*, \mathbf{1}) \\ &\cong \text{Hom}(X_1 \boxtimes X_2, (V^{**} \boxtimes \mathbf{1}) \otimes A) \end{aligned}$$

for all objects $X_1, X_2 \in \mathcal{C}$, where the two isomorphisms in the middle come from the braiding in \mathcal{C} . On the other hand the isomorphism $(u_V \boxtimes \text{id}_1) \otimes \text{id}_A$ is given by

$$\begin{aligned} \text{Hom}(X_1 \boxtimes X_2, (V \boxtimes \mathbf{1}) \otimes A) &\cong \text{Hom}(V^* \otimes X_1 \otimes X_2, \mathbf{1}) \\ &\cong \text{Hom}(X_1 \otimes X_2 \otimes V^*, \mathbf{1}) \\ &\cong \text{Hom}(X_1 \boxtimes X_2, (V^{**} \boxtimes \mathbf{1}) \otimes A), \end{aligned}$$

therefore, the hexagon identity and Proposition 2.3(a) imply that

$$\iota_V = (u_V \boxtimes \text{id}_1) \otimes \text{id}_A.$$

Next, using the definition of ι_V in (19) and the naturality of braiding, we compute:

$$\begin{aligned} \iota_{**V}^{-1} \circ \iota_V^* &= \\ &= (\rho_{**V}^{-1} \circ \Sigma_{\mathbf{1} \boxtimes {}^{**}V, A}^{-1} \circ (\rho_V^*)^{-1} \circ \Sigma_{A, V \boxtimes \mathbf{1}}^{-1}) \circ (\rho_V^* \circ \Sigma_{\mathbf{1} \boxtimes {}^{**}V, A} \circ \rho_V^{**} \circ \Sigma_{A, V^{**} \boxtimes \mathbf{1}}) \\ &= \rho_{**V}^{-1} \circ \Sigma_{\mathbf{1} \boxtimes {}^{**}V, A}^{-1} \circ (\rho_V^*)^{-1} \circ (\rho_V^* \circ \Sigma_{\mathbf{1} \boxtimes {}^{**}V, A} \circ \rho_V^{**} \circ \Sigma_{A, V^{**} \boxtimes \mathbf{1}}) \circ \Sigma_{A, V^{**} \boxtimes \mathbf{1}}^{-1} \\ &= \rho_{**V}^{-1} \circ \rho_V^{**}. \end{aligned}$$

On the other hand,

$$\iota_{**V}^{-1} \circ \iota_{*V}^* = (u_{**V}^{-1} \circ u_{*V}^* \boxtimes \text{id}_{\mathbf{1}}) \otimes \text{id}_A,$$

therefore Proposition 2.3(a) and equation (17) imply the result. \square

6. COMPARISON WITH [EO]

In this Section we show that our Definition 3.1 of a distinguished invertible object D of \mathcal{C} agrees with [EO, Definition 2.12].

Recall that it was proved in [EO] that in a finite tensor category any projective object is injective and vice versa. In particular the projective cover P_0 of the unit object $\mathbf{1} \in \mathcal{C}$ coincides with the injective hull of some object $\tilde{D} \in \mathcal{C}$. It was shown in [EO] that \tilde{D} is an invertible object.

Theorem 6.1. *The object \tilde{D} is isomorphic to D .*

Proof. Let I be a set indexing the isomorphism classes of simple objects in \mathcal{C} ; for $\alpha \in I$ let $L_\alpha, P_\alpha, I_\alpha$ denote a simple object corresponding to α , its projective cover, and its injective hull. We will assume that $0 \in I$ and $L_0 = \mathbf{1}$. Let i runs through I . We are going to compute $\dim \text{Hom}(P_0 \boxtimes L_i, A^*)$ in two ways.

First calculation:

$$\begin{aligned} \dim \text{Hom}(P_0 \boxtimes L_i, A^*) &= \dim \text{Hom}(P_0 \boxtimes L_i, (D \boxtimes \mathbf{1}) \otimes A) \\ &= \dim \text{Hom}(P_0 \otimes L_i, D) = \begin{cases} 1 & \text{if } L_i = D, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Second calculation:

$$\begin{aligned} \dim \text{Hom}(P_0 \boxtimes L_i, A^*) &= \dim \text{Hom}(A, {}^*(P_0 \boxtimes L_i)) \\ &= \dim \text{Hom}((P_0 \boxtimes \mathbf{1}) \otimes A, \mathbf{1} \boxtimes L_i^*). \end{aligned}$$

Let us look closely at the object $(P_0 \boxtimes \mathbf{1}) \otimes A$ in $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$.

Lemma 6.2. *The object $(P_0 \boxtimes \mathbf{1}) \otimes A$ is injective.*

Proof. Observe that the functor

$$\text{Hom}(X \boxtimes Y, (P_0 \boxtimes \mathbf{1}) \otimes A) = \text{Hom}((P_0^* \otimes X) \boxtimes Y, A) = \text{Hom}(P_0^* \otimes X \otimes Y, \mathbf{1})$$

is exact in both variables X, Y since $P_0^* \otimes X \otimes Y$ is injective, see [EO]. Thus the functor $\text{Hom}(\cdot, (P_0 \boxtimes \mathbf{1}) \otimes A)$ is exact, see [D]. The Lemma is proved. \square

We continue the proof of the Theorem. By Lemma 6.2,

$$(P_0 \boxtimes \mathbf{1}) \otimes A = \sum_{\alpha, \beta \in I} M_{\alpha\beta} I_\alpha \boxtimes I_\beta$$

for some non-negative integer multiplicities $M_{\alpha\beta}$. We have

$$\begin{aligned} M_{\alpha\beta} &= \dim \text{Hom}(L_\alpha \boxtimes L_\beta, (P_0 \boxtimes \mathbf{1}) \otimes A) \\ &= \dim \text{Hom}(P_0^* \otimes L_\alpha \otimes L_\beta, \mathbf{1}) \\ &= \dim \text{Hom}(L_\alpha \otimes L_\beta, P_0) \\ &= [L_\alpha \otimes L_\beta : \tilde{D}], \end{aligned}$$

where $[X : L_i]$ denotes the multiplicity of a simple object L_i in the Jordan-Hölder series of X . To calculate $\dim \text{Hom}((P_0 \boxtimes \mathbf{1}) \otimes A, \mathbf{1} \boxtimes L_i^*)$ it is enough to consider

the summands with $I_\alpha = P_0$. In this case $L_\alpha = \tilde{D}$ and $[L_\alpha \otimes L_\beta : \tilde{D}] = [L_\beta : \mathbf{1}]$. Thus

$$(20) \quad \dim \operatorname{Hom}((P_0 \boxtimes \mathbf{1}) \otimes A, \mathbf{1} \boxtimes L_i^*) = \begin{cases} 1 & \text{if } I_0 \text{ covers } L_i^*, \\ 0 & \text{otherwise.} \end{cases}$$

Since $I_0 = P_0^*$ covers \tilde{D}^{-1} the Theorem follows. \square

Remark 6.3. Note that Theorems 3.3 and 6.1 together are equivalent to [EO, Conjecture 2.15].

Corollary 6.4. *A semisimple finite tensor category is unimodular.*

Here is another application of Lemma 6.2 (which in the case of Hopf algebras is a well-known statement).

Proposition 6.5. *Let $f : A \rightarrow A^{**}$ be a morphism in $\mathcal{C} \boxtimes \mathcal{C}^{op}$. Assume that $\operatorname{Tr}(f) \neq 0$. Then the category \mathcal{C} is semisimple.*

Proof. By definition $\operatorname{Tr}(f)$ is the following morphism :

$$(21) \quad \operatorname{Tr}(f) : \mathbf{1} \boxtimes \mathbf{1} \xrightarrow{\operatorname{coev}_A} A \otimes A^* \xrightarrow{f \otimes \operatorname{id}_A} A^{**} \otimes A^* \xrightarrow{\operatorname{ev}_{A^*}} \mathbf{1} \boxtimes \mathbf{1}.$$

In particular, if $\operatorname{Tr}(f) \neq 0$ then $\mathbf{1}$ is a direct summand of $A \otimes A^*$. Hence $P_0 \boxtimes \mathbf{1}$ is a direct summand of $(P_0 \boxtimes \mathbf{1}) \otimes A \otimes A^*$. By Lemma 6.2 $(P_0 \boxtimes \mathbf{1}) \otimes A$ is projective and therefore $(P_0 \boxtimes \mathbf{1}) \otimes A \otimes A^*$ is projective. Thus $P_0 \boxtimes \mathbf{1}$ is projective and consequently $\mathbf{1}$ is projective. Hence \mathcal{C} is semisimple. \square

7. FUSION CATEGORIES

In this section we specialize the previous considerations to the case when the category \mathcal{C} is semisimple (and thus \mathcal{C} is a fusion category, see [ENO]). Let $\{L_i\}_{i \in I}$ be a set of representatives of isomorphism classes of simple objects in \mathcal{C} .

Let us describe the structure of the algebra A . It follows immediately from definitions that we have a canonical isomorphism

$$(22) \quad A = \bigoplus_{i \in I} L_i \boxtimes {}^* L_i.$$

Using the definitions one describes the multiplication in the algebra A in the following way (cf. [Mu]): we have $A \otimes A = \bigoplus_{i,j \in I} (L_i \otimes L_j) \boxtimes {}^*(L_i \otimes L_j)$; for any $m \in I$ the vector spaces $\operatorname{Hom}(L_i \otimes L_j, L_m)$ and $\operatorname{Hom}({}^*(L_i \otimes L_j), {}^* L_m) = \operatorname{Hom}(L_m, L_i \otimes L_j)$ are canonically dual to each other via the pairing

$$\operatorname{Hom}(L_i \otimes L_j, L_m) \otimes \operatorname{Hom}(L_m, L_i \otimes L_j) \rightarrow \operatorname{Hom}(L_m, L_m) = k$$

and hence there is a canonical morphism $(L_i \otimes L_j) \boxtimes {}^*(L_i \otimes L_j) \rightarrow L_m \boxtimes {}^* L_m$. Then the multiplication $A \otimes A \rightarrow A$ is just a direct sum (over $m \in I$) of all such morphisms.

Next, we describe the canonical isomorphism $A \cong A^{**}$. We have canonically

$$A^{**} \cong \bigoplus_{i \in I} L_i^{**} \boxtimes {}^{***} L_i \cong \bigoplus_{i \in I} L_i \boxtimes {}^{*****} L_i.$$

Thus to specify a morphism $A \rightarrow A^{**}$ in $\mathcal{C} \boxtimes \mathcal{C}^{op}$ is the same as to specify a collection of morphisms $\psi_i : {}^* L_i \rightarrow {}^{*****} L_i$ in \mathcal{C} . Recall [Mu, ENO] that for any simple object L in \mathcal{C} one defines its *squared norm* $|L|^2$ as follows: choose an isomorphism $\phi : L \rightarrow L^{**}$ (such an isomorphism always exists and is unique up to a scaling) and

set $|L|^2 = \text{Tr}(\phi)\text{Tr}((\phi^{-1})^*)$, where Tr is defined as in equation (21). Note that $|L|^2$ does not change after rescaling the evaluation and coevaluation morphisms in \mathcal{C} .

Lemma 7.1. *The canonical isomorphism $A \rightarrow A^{**}$ corresponds to the collection of morphisms $\psi_i : {}^*L_i \rightarrow {}^{****}L_i$ characterized by the following property: for any isomorphism $\phi_i : {}^*L_i \rightarrow {}^{***}L_i$ one has $\text{Tr}(\phi_i^{-1})\text{Tr}(\phi_i \circ \psi_i^{-1}) = |L_i|^2$.*

Proof. The statement is immediate from definitions since the isomorphism $A \cong A^{**}$ is the composition of the isomorphism $A \cong A^*$ and of the inverse of the dual of this isomorphism. \square

Corollary 7.2. *The trace of the canonical isomorphism $A \cong A^{**}$ is equal to $\dim(\mathcal{C}) := \sum_{i \in I} |L_i|^2$.*

Let us relate the canonical isomorphism $\delta : ?^{**} \cong {}^{**}?$ from Theorem 3.3 with squared norms of simple objects of \mathcal{C} .

Theorem 7.3. *Let $L \in \mathcal{C}$ be a simple object. The canonical isomorphism $\delta_L : L^{**} \cong {}^{**}L$ can be characterized in the following way: for any isomorphism $\phi : L^{**} \rightarrow L$ one has $\text{Tr}(\phi^{-1})\text{Tr}(\phi \circ \delta_L^{-1}) = |L_i|^2$.*

Proof. Recall that A represents the functor $X \boxtimes Y \mapsto \text{Hom}(X \otimes Y, \mathbf{1})$ and A^{**} represents the functor $X \boxtimes Y \mapsto \text{Hom}({}^{**}X \otimes Y^{**}, \mathbf{1}) = \text{Hom}(X \otimes Y^{****}, \mathbf{1})$. It follows immediately from definitions that the canonical isomorphism $A \rightarrow A^{**}$ corresponds to the natural transformation $X \boxtimes Y \xrightarrow{\text{id} \boxtimes \delta_{Y^{**}}^{-1}} X \boxtimes Y^{****}$. Now the Theorem is an immediate consequence of Lemma 7.1. \square

Corollary 7.4. *Let $V \in \mathcal{C}$ be an object and let $\phi : V \rightarrow {}^{**}V$ be a morphism. Then $\text{Tr}(\phi^*) = \text{Tr}(\phi \circ \delta_V^{-1})$.*

Proof. It is enough to prove the statement for $V = L_i$ and any isomorphism $\phi : L_i \rightarrow {}^{**}L_i$. But this is an immediate consequence of Theorem 7.3. \square

Remark 7.5. The following example shows that the statement of Corollary 7.4 is not true if \mathcal{C} is only assumed to be unimodular. Let q be a primitive p th root of unity and let $U_q(sl_2)$ be the corresponding finite dimensional quantum sl_2 Hopf algebra. Let $H = \text{gr}(U_q(sl_2))$ be the associated graded Hopf algebra of $U_q(sl_2)$. This is a Hopf algebra of dimension p^3 defined like $U_q(sl_2)$ except that $EF - FE = 0$. Since $U_q(sl_2)$ is unimodular, so is $\text{gr}(U_q(sl_2))$. Then the statement of Corollary 7.4 would say that $\text{Tr}(K) = \text{Tr}(K^{-1})$ in any finite dimensional representation, which is false in 1-dimensional representations.

Recall (see [ENO]) that for any fusion category \mathcal{C} one constructs a twice bigger category $\tilde{\mathcal{C}}$ called its *pivotalization*. By definition, the objects of $\tilde{\mathcal{C}}$ are pairs (X, f) where X is an object of \mathcal{C} and $f : X \rightarrow X^{**}$ is an isomorphism satisfying $f^{**}f = d_{X^{**}}$. It is easy to see that the category $\tilde{\mathcal{C}}$ has a canonical pivotal structure.

Corollary 7.6. *The category $\tilde{\mathcal{C}}$ is spherical, that is $\dim(X) = \dim(X^*)$ for any $X \in \tilde{\mathcal{C}}$.*

Proof. Follows from Corollary 7.4. \square

Remark 7.7. The statement of corollary was proved in [ENO] under assumption that $\dim(\mathcal{C}) \neq 0$, which is automatically satisfied in characteristic zero. Thus our result is new only in positive characteristic.

In the case when \mathcal{C} is the representation category of a semisimple Hopf algebra H , Corollary 7.4 becomes the following statement, which appears to be new in the case of positive characteristic (however, see [LR2, Theorem 2.2] and [R4, Propositions 9 and 10] for related statements).

Proposition 7.8. *If H is a semisimple Hopf algebra over an algebraically closed field k and g the distinguished grouplike element in H (so that $g^{-1}xg = S^4(x)$). Then for any element $a \in H$ such that $axa^{-1} = S^2(x)$ and any irreducible H -module V , one has*

$$(23) \quad \text{Tr}_V(a) = \text{Tr}_V(ga).$$

Remark 7.9. Let us give another, purely Hopf-algebraic proof of this statement. Consider a new Hopf algebra K obtained by extending H by a grouplike element a such that $a^2 = g^{-1}$ and $axa^{-1} = S^2(x)$ [So]. It is well known that S^2 is an inner automorphism of H , so every irreducible representation V of H extends to a representation of K (in two ways). We must show that $\text{Tr}_V(a) = \text{Tr}_V(ga)$ for either of these extensions.

To do this, we recall that the left integral in K^* is given by the formula [LR1, Proposition 2.4(a)]

$$\lambda(x) = \sum_{W \in \text{Irrep} K} \text{Tr}_W(xa) \text{Tr}_W(a^{-1}).$$

Also, recall that under right multiplication in K^* , λ changes according to the character g . So for any $f \in K^*$ we get

$$\sum_W \text{Tr}_W(x_{(1)}a) f(x_{(2)}) \text{Tr}_W(a^{-1}) = \sum_W \text{Tr}_W(xa) \text{Tr}_W(a^{-1}) f(g).$$

(we use the Sweedler notation $\Delta(z) = z_{(1)} \otimes z_{(2)}$, implying the summation as usual). Set $f(z) = \text{Tr}_V(za)$, and $x = I$ (the integral of K acting by 1 in the trivial module; it exists since K is semisimple). Then we get

$$\sum_W \text{Tr}_W(I_{(1)}a) \text{Tr}_V(I_{(2)}a) \text{Tr}_W(a^{-1}) = \sum_W \text{Tr}_W(Ia) \text{Tr}_W(a^{-1}) \text{Tr}_V(ga).$$

The right hand side of this equation is obviously equal to $\text{Tr}_V(ga)$ (only the term with $W = k$ survives). As to the left hand side, it can be written as

$$\sum_W \text{Tr}_{W \otimes V}(Ia) \text{Tr}_W(a^{-1}) = \sum_W \text{Tr}_{W \otimes V}(I) \text{Tr}_W(a^{-1}).$$

Here the only nonzero summand comes from $W = V^*$, and it yields $\text{Tr}_{V^*}(a^{-1}) = \text{Tr}_{V^*}(S(a)) = \text{Tr}_V(a)$. Thus, $\text{Tr}_V(ga) = \text{Tr}_V(a)$, and we are done.

Corollary 7.10. *If H is a semisimple Hopf algebra over k and $S^4 = 1$ in H then H^* is unimodular.*

Proof. In this case g is central, so we get $\text{Tr}_V(a) = g_V \text{Tr}_V(a)$, where g_V is the eigenvalue of g on V . But $\text{Tr}_V(a)$ is nonzero, so $g_V = 1$ and hence $g = 1$. Thus H^* is unimodular. \square

Remark 7.11. In the case $S^2 = 1$, this result is contained in [L].

8. APPENDIX

We will use the notation of [EO]. Let \mathcal{C} be a finite tensor category, and \mathcal{M} be an abelian category which carries the structure of a module category over \mathcal{C} . Let B be a finite dimensional algebra over k , and suppose that an equivalence of categories $\mathcal{M} \rightarrow B\text{-mod}$ is fixed. In this case, any object $X \in \mathcal{C}$ defines an exact functor $X \otimes : B\text{-mod} \rightarrow B\text{-mod}$. This means that we have a tensor functor $F : \mathcal{C} \rightarrow B\text{-bimod}$, such that $X \otimes M = F(X) \otimes_B M$ for $M \in B\text{-Mod}$.

Let $\bar{F} := F \circ \text{Forget} : \mathcal{C} \rightarrow \text{Vect}_k$ be the fiber functor, and $H := \text{End} \bar{F}$. Thus $\mathcal{C} = H\text{-mod}$ as an abelian category. (In fact, H has the structure of a Hopf algebroid, reflecting the fact that \mathcal{C} is a finite tensor category). In particular, we have a homomorphism $\mu : B \otimes B^\circ \rightarrow H$, coming from the functor F . Thus H is a module over $B \otimes B^\circ$, where B° is the algebra opposite to B , via $a \circ h := \mu(a)h$ for all $h \in H$ and $a \in B \otimes B^\circ$.

Recall [EO] that \mathcal{M} is called an *exact* module category if for any projective object $P \in \mathcal{C}$ and any $M \in \mathcal{M}$, the object $P \otimes M$ is projective. Equivalently, \mathcal{M} is exact if any \mathcal{C} -module additive functor from \mathcal{M} is exact.

The main result of this appendix is the following algebraic characterization of the property of exactness.

Theorem 8.1. *\mathcal{M} is exact if and only if H is a projective $B \otimes B^\circ$ -module.*

Proof. Recall that a module V over a finite dimensional algebra A is projective if and only if it is flat. Indeed, $V \otimes_A W = \text{Hom}_A(V, W^*)^*$ for any two finite dimensional left A -modules V, W , and hence the functor $V \otimes_A ?$ is exact iff so is $\text{Hom}_A(V, ?)$.

Suppose \mathcal{M} is an exact module category over \mathcal{C} . We see that to prove the required statement, it suffices to show that the module H over $B \otimes B^\circ$ is flat.

Since H is a left H -module, we can regard H as an object of \mathcal{C} . Thus, the functor $H \otimes_B ?$ is exact, and for any finite dimensional left B -module M , the module $H \otimes_B M$ is projective.

To show that H is flat, let M be a left B -module, N a right B -module, and let us compute the derived tensor product $H \otimes_{B \otimes B^\circ}^L (M \otimes N)$. By the Künneth formula we have

$$H \otimes_{B \otimes B^\circ}^L (M \otimes N) \cong (H \otimes_B^L M) \otimes_{B^\circ}^L N.$$

Now, since the functor $H \otimes_B$ is exact, we have

$$(H \otimes_B^L M) \otimes_{B^\circ}^L N \cong (H \otimes_B M) \otimes_{B^\circ}^L N.$$

Further, since the module $H \otimes_B M$ is projective, it is also flat. Thus,

$$(H \otimes_B M) \otimes_{B^\circ}^L N \cong (H \otimes_B M) \otimes_{B^\circ} N.$$

We conclude that

$$H \otimes_{B \otimes B^\circ}^L (M \otimes N) \cong H \otimes_{B \otimes B^\circ} (M \otimes N),$$

which implies that H is flat, as desired.

Conversely, assume that H is a projective module. Then $H \cong \oplus_{i,j} V_{ij} \otimes P_i \otimes P_j^\circ$, where V_{ij} are vector spaces, and P_i, P_j° are the projective covers of irreducible modules over B, B° , respectively. Thus, for any left B -module M we have

$$H \otimes_B M = \oplus_i [\oplus_j V_{ij} \otimes (P_j^\circ \otimes_B M)] \otimes P_i.$$

This B -module is obviously projective, so \mathcal{M} is exact and we are done. \square

REFERENCES

- [BK] B. Bakalov, A. Kirillov Jr., *Lectures on Tensor categories and modular functors*, AMS, Providence, (2001).
- [BNS] G. Böhm, F. Nill, and K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^* -structure*, J. Algebra, **221** (1999), 385–438.
- [BT] D. Bulacu, B. Torrecillas, *Factorizable quasi-Hopf algebras. Applications*, math.QA/0312076 (2003).
- [D] P. Deligne, *Catégories tannakiennes*, in : *The Grothendieck Festschrift*, Vol. II, Progr. Math., **87**, Birkhuser, Boston, MA, 1990, 111–195.
- [ENO] P. Etingof, D. Nikshych, V. Ostrik, *On fusion categories*, math.QA/0203060 (2002).
- [EO] P. Etingof, V. Ostrik, *Finite tensor categories*, math.QA/0301027 (2003).
- [HN] F. Hausser, F. Nill, *Integral theory for quasi-Hopf algebras*, math.QA/9904164 (1999).
- [K] C. Kassel, *Quantum groups*, Graduate texts in mathematics **155**, Springer-Verlag, New York (1995).
- [Ku] G. Kuperberg, *Noninvolutory Hopf algebras and 3-manifold invariants*, Duke Math. J. **84** (1996), no. 1, 83–129.
- [L] R.G. Larson, *Characters of Hopf algebras*, J. of Algebra **17** (1971) 352–368.
- [LR1] R.G. Larson, D. Radford, *Finite dimensional semisimple Hopf algebras in characteristic 0 are semisimple*, J. of Algebra, **117** (1988) 267–289.
- [LR2] R.G. Larson, D. Radford, *Semisimple Hopf Algebras*, J. of Algebra, **172** (1995) 5–35.
- [LS] R.G. Larson, M.E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, American J. of Math., **91** (1969), 75–94.
- [L] G. Lusztig, *Quantum groups at root of 1*, Geom. Dedicata, **35** (1990), 89–113.
- [M] S. MacLane, *Categories for the working mathematician*, Graduate texts in mathematics, **5**, 2nd edition, Springer-Verlag (1998).
- [Mu] M. Müger, *From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Algebra **180** (2003), 81–157.
- [N] D. Nikshych, *On the structure of weak Hopf algebras*, Adv. Math., **170** (2002), 257–286.
- [NTV] D. Nikshych, V. Turaev, and L. Vainerman, *Invariants of knots and 3-manifolds from quantum groupoids*, Topology and its Applications, **127** (2003), 91–123.
- [OI] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups, **8** (2003), 177–206.
- [R1] D. Radford, *The order of the antipode of a finite dimensional Hopf algebra is finite*, Amer. J. Math. **98** (1976), 333–355.
- [R2] D. Radford, *On the antipode of a quasitriangular Hopf algebra*, J. Algebra, **151** (1992), 1–11.
- [R3] D. Radford, *On Kauffman’s knot invariants arising from finite-dimensional Hopf algebras*, in : *Advances in Hopf algebras*, Lecture Notes in Pure and Appl. Math., **158** (1994), 205–266.
- [R4] D. Radford, *The trace function and Hopf algebras*, J. of Algebra, **163** (1994), 583–622.
- [RS] N. Reshetikhin, M. Semenov-Tian-Shansky, *Quantum R-matrices and factorization problems*, J. Geom. Phys., **5** (1988), 533–550.
- [S] H.-J. Schneider, *Some properties of factorizable Hopf algebras*, Proc. Amer. Math. Soc., **129** (2001), 1891–1898.
- [So] Y. Sommerhäuser, *On Kaplansky’s fifth conjecture*, J. Algebra **204** (1998), 202–224.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, USA

E-mail address: etingof@math.mit.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NH 03824, USA

E-mail address: nikshych@math.unh.edu

INSTITUTE FOR ADVANCED STUDY, EINSTEIN DR, PRINCETON, NJ 08540, USA

E-mail address: ostrik@math.mit.edu